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The Noether Conservation Laws of Some Vaidiya Metrics

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Abstract In this paper, we show that a large amount information can be extracted from a knowledge of the vector fields that leave the action integral invariant, viz., Noether symmetries. In addition to a larger class of conservation laws than those given by the isometries or Killing vectors, we may conclude what the isometries are and that these form a Lie subalgebra of the Noether symmetry algebra. We perform our analysis on versions of the Vaidiya metric yielding some previously unknown information regarding the corresponding manifold. Lastly, with particular reference to this metric, we show that the only variations on $m(u)$ that occur are $m = 0$, $m =$ constant, $m = u$ and $m = m(u)$.

Keywords Vaidiya metric · Conservation laws · Noether symmetries

1 Introduction

In the paper [1] by Lindquist et al., a detailed description on why the Vaidiya metric [2]

$$\mathrm{d}s^2 = -\left[1 - \frac{2m(u)}{r}\right]\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{1.1}$$

where $m(u)$ is the arbitrary function of the retarded time coordinate u, is the most convenient one for the spherically symmetric solution of the Einstein equations with the geometrical optics stress energy tensor of radiation. Following the coordinates introduced by Finkelstein [3], the metric in (1.1) can be construed as

$$\mathrm{d}s^2 = -\left[1 - \frac{2m(u)}{r}\right]\mathrm{d}u^2 - 4\frac{m(u)}{r}\mathrm{d}u\mathrm{d}r + \left[1 + \frac{2m(u)}{r}\right]\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1.2}$$

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Various studies relating to the Vaidiya metric has been done, for e.g., the 'nature of naked singularities' [4], the 'Carter constant and Petrov classification' [5] and references therein which include aspects of the nature of the Killing tensors and well known notion of the 'isometries' of the metric which are the *diffeomorphisms of the manifold onto itself which preserve the metric tensor* [7].

In this paper, we consider a novel approach to the 'invariance' studies associated with the metric in (1.1) and (1.2). We show that we totally recover the information regarding the isometries of the metric from a study of the Noether symmetries associated with the corresponding natural Lagrangian, L, which preserve the action $\mathcal{L} = \int L$ and more. That is, a larger algebra of generators of symmetry are obtained and, hence, more conservation laws classified according to the function $m(u)$. For the cases of the first metric (1.1), we also determine the Lie algebra of symmetry generators of the geodesic equations (Euler-Lagrange equations) which contains all the above as subalgebras.

We briefly state some of the features of an Euler-Lagrange system of differential equations (des). Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz.,

$$E^{\beta}(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \beta = 1, \dots, \tilde{m}. \tag{1.3}$$

A conservation law of (1.3) is the equation

$$D_i T^i = 0, \tag{1.4}$$

on the solutions of (1.3). Here the *total differentiation operator* is

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \cdots, \quad i = 1, \dots, n.$$

The tuple $T = (T^1, \dots, T^n)$ is called a *conserved vector* of (1.3).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie-Bäcklund operator (vector field) is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots, \tag{1.5}$$

where $\xi^i, \eta^{\alpha} \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \\ &\vdots \end{aligned} \tag{1.6}$$

and W^{α} is the Lie characteristic function defined by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{1.7}$$

In this paper, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u.

The Euler-Lagrange equations, if they exist, associated with (1.3) is the system $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{1.8}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(x, u, u_{(1)}, \ldots, u_{(r)})\mathrm{d}x \tag{1.9}$$

defined over Ω. If we include point dependent gauge terms $g_1, \ldots, g_n$, the Noether symmetries X are given by

$$XL + LD_i\xi_i = D_i g_i. \tag{1.10}$$

Corresponding to each X, a conserved vector $T = (T^1, \ldots, T^n)$ is obtained via Noether's Theorem.

A Lie symmetry generator of (1.3) is a one parameter Lie group transformation (vector field) that leave the system invariant modulo the solutions of (1.3).

2 Symmetries of the Metric (1.1)

The Euler-Lagrange (geodesic) equations associated with the Lagrangian

$$L = -\left(1 - \frac{2m(u)}{r}\right)\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{2.1}$$

corresponding to (1.1) is

$$\begin{aligned}
\ddot{u} &= \frac{-r^3\dot{\phi}^2 + \cos\theta^2 r^3\dot{\phi}^2 - r^3\dot{\theta}^2 + m\dot{u}^2}{r^2},\\
\ddot{r} &= \frac{-2m\dot{r}\dot{u} + 2mr\ddot{u} - r^2\ddot{u}}{r^2},\\
\ddot{\theta} &= \frac{r\dot{\phi}^2\cos\theta\sin\theta - 2\dot{r}\dot{\theta}}{r},\\
\ddot{\phi} &= \frac{-2r\dot{\phi}\dot{\theta}\csc\theta\cos\theta - 2\dot{r}\dot{\phi}}{r\csc\theta\sin\theta},
\end{aligned} \tag{2.2}$$

where $\dot{\alpha}$ is the derivative of α with respect to the arclength parameter s.

(I) *Lie point symmetries.*

The algebra of Lie point symmetries of (2.2) separate into a number of classes based on $m(u)$. The ivaraince of differential equations leading to Lie symmetries is now well documented and can be found in, inter alia, [6]. We list the following two classes. We note the 'large' 35-dimensional Lie algebra for the first case reduces radically when we consider the Noether symmetries (later).

(i) Case $m = 0$:

$$
\begin{aligned}
X_1 &= \partial_\phi,\\
X_2 &= \partial_s,\\
X_3 &= s\partial_s,\\
X_4 &= (r+u)\partial_s,\\
X_5 &= r\cos\theta\partial_s,\\
X_6 &= r\cos\phi\sin\theta\partial_s,\\
X_7 &= r\sin\phi\sin\theta\partial_s,\\
X_8 &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi,\\
X_9 &= \sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi,\\
X_{10} &= \partial_u,\\
X_{11} &= s\partial_u,\\
X_{12} &= (r+u)\partial_u,\\
X_{13} &= r\cos\theta\partial_u,\\
X_{14} &= r\cos\phi\sin\theta\partial_u,\\
X_{15} &= r\sin\phi\sin\theta\partial_u,\\
X_{16} &= u\partial_u + r\partial_r,\\
X_{17} &= s^2\partial_s + su\partial_u + rs\partial_r,\\
X_{18} &= s(r+u)\partial_s + u(r+u)\partial_u + r(r+u)\partial_r,\\
X_{19} &= \cos\theta\partial_u - \cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta,\\
X_{20} &= s\cos\theta\partial_u - s\cos\theta\partial_r + \frac{s}{r}\sin\theta\partial_\theta,\\
X_{21} &= u\cos\theta\partial_u - (r+u)\cos\theta\partial_r + \frac{(r+u)}{r}\sin\theta\partial_\theta,\\
X_{22} &= rs\cos\theta\partial_s + ru\cos\theta\partial_u + r^2\cos\theta\partial_r,\\
X_{23} &= u + r\cos^2 t\partial_u + r\sin^2 t\partial_r + \cos\theta\sin\theta\partial_\theta,\\
X_{24} &= s\cos\phi\sin\theta\partial_u - s\cos\phi\sin\theta\partial_r - \frac{s}{r}\cos\phi\cos\theta\partial_\theta + \frac{s}{r}\csc\theta\sin\theta\partial_\phi,\\
X_{25} &= u\cos\phi\sin\theta\partial_u - (r+u)\cos\phi\cos\theta\partial_r - \frac{(r+u)}{r}\cos\phi\cos\theta\partial_\theta\\
&\quad + \frac{(r+u)}{r}\csc\theta\sin\theta\partial_\phi,\\
X_{26} &= rs\cos\phi\sin\theta\partial_s + ru\cos\phi\sin\theta\partial_u + r^2\cos\phi\sin\theta\partial_r,\\
X_{27} &= r\cos\phi\sin 2\theta\partial_u - 2r\cos\phi\cos\theta\sin\theta\partial_r - \cos\phi\cos 2\theta\partial_\theta + \cot\theta\sin\phi\partial_\phi,
\end{aligned}
$$

$$X_{28} = s\sin\phi\sin\theta\partial_u - s\sin\phi\sin\theta\partial_r - \frac{s}{r}\cos\theta\sin\phi\partial_\theta - \frac{s}{r}\cos\phi\csc\theta\partial_\phi,$$

$$X_{29} = u\sin\phi\sin\theta\partial_u - (r+u)\sin\phi\sin\theta\partial_r - \frac{(r+u)}{r}\cos\theta\sin\phi\partial_\theta - \frac{(r+u)}{r}\cos\phi\csc\theta\partial_\phi,$$

$$X_{30} = rs\sin\phi\sin\theta\partial_s + ru\sin\phi\sin\theta\partial_u + r^2\sin\phi\sin\theta\partial_r,$$

$$X_{31} = r\sin\phi\sin 2\theta\partial_u - r\sin\phi\sin 2\theta\partial_r - \cos 2\theta\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi,$$

$$X_{32} = 2r\cos 2\phi\sin^2\theta\partial_u - 2r\cos 2\phi\sin^2\theta\partial_r - \cos 2\phi\sin 2\theta\partial_\theta + 2\sin 2\phi\partial_\phi,$$

$$X_{33} = 4r\cos\phi\sin\phi\sin^2\theta\partial_u - 4r\cos\phi\sin\phi\sin^2\theta\partial_r - \sin 2\phi\sin 2\theta\partial_\theta - 2\cos 2\phi\partial_\phi,$$

$$X_{34} = -\cos\phi\sin\theta\partial_u + \cos\phi\sin\theta\partial_r + \frac{1}{r}\cos\phi\cos\theta\partial_\theta - \frac{1}{r}\sin\phi\csc\theta\partial_\phi,$$

$$X_{35} = -\sin\phi\sin\theta\partial_u + \sin\phi\sin\theta\partial_r + \frac{1}{r}\sin\phi\cos\theta\partial_\theta - \frac{1}{r}\cos\phi\csc\theta\partial_\phi.$$

(ii) Case $m = k$:

$$\begin{aligned}
X_1 &= \partial_\phi,\\
X_2 &= \partial_s,\\
X_3 &= s\partial_s,\\
X_4 &= \partial_u,\\
X_5 &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi,\\
X_6 &= -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi,
\end{aligned}$$

which yields the following commutator table

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-X_5$	X_4	0
X_2	0	0	X_2	0	0	0
X_3	0	$-X_2$	0	0	0	0
X_4	X_5	0	0	0	$-X_1$	0
X_5	$-X_4$	0	0	X_1	0	0
X_6	0	0	0	0	0	0

(II) *Noether symmetries*.

The Lagrangian (2.1), viz., $L = -(1 - \frac{2m(u)}{r})\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$, leads to the following Noether symmetries from (1.10).

(i) Case $m = 0$:

$$X_1 = -\frac{1}{2}s^2\partial_s - \frac{1}{2}su\partial_u - \frac{1}{2}rs\partial_r, \quad g = \frac{1}{2}u^2,$$

$$X_2 = -\frac{1}{2}s\partial_u, \quad g = \frac{1}{2}u + r,$$

$$X_3 = -\frac{1}{2}\sin\theta\sin\phi\partial_u + \frac{1}{2}(s\csc\theta\sin\phi - \tan\theta\cos\theta\sin\phi)\partial_r$$
$$+ \frac{1}{2}\frac{s}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{2}\frac{s}{r}\csc\theta\cos\phi\partial_\phi, \quad g = r\sin\theta\sin\phi,$$
$$X_4 = -\frac{1}{2}\sin\theta\cos\phi\partial_u + \frac{1}{2}(s\csc\theta\cos\phi - \tan\theta\cos\theta\cos\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\cos\phi\partial_\theta$$
$$- \frac{1}{2}\frac{s}{r}\csc\theta\sin\phi\partial_\phi, \quad g = r\sin\theta\cos\phi,$$
$$X_5 = -\frac{1}{2}s\cos\theta\partial_u + \frac{1}{2}s\cos\theta\partial_r - \frac{1}{2}\frac{s}{r}\sin\theta\partial_\theta, \quad g = r\cos\theta,$$
$$X_6 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = 0,$$
$$X_7 = \partial_s, \quad g = 0,$$
$$X_8 = u\sin\theta\sin\phi\partial_u + \csc\theta(-u\sin\phi - 2r\sin^2\theta\sin\phi + 2u\cos^2\theta\sin\phi)\partial_r$$
$$+ \left(\frac{u}{r}\cos\theta\sin\phi + \tan\theta\sin\theta\sin\phi - \sec t\sin\phi\right)\partial_\theta$$
$$- \left(\frac{u}{r} + 1\right)\csc\theta\cos\phi\partial_\phi, \quad g = 0,$$
$$X_9 = u\sin\theta\cos\phi\partial_u + (u\tan\theta\cos\theta\cos\phi - r\sin\theta\cos\phi - u\csc\theta\cos\phi)\partial_r$$
$$+ \left(\tan\theta\sin\theta\cos\phi - \sec\theta\cos\phi - \frac{u}{r}\cos\theta\cos\phi\right)\partial_\theta$$
$$+ \left(\frac{u}{r} + 1\right)\csc\theta\sin\phi\partial_\phi, \quad g = 0,$$
$$X_{10} = u\cos\theta\partial_u + (-r\cos\theta - u\cos\theta)\partial_r + \left(\frac{u}{r} + 1\right)\sin\theta\partial_\theta, \quad g = 0,$$
$$X_{11} = \partial_u, \quad g = 0,$$
$$X_{12} = \cos\theta\partial_u - \cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta, \quad g = 0,$$
$$X_{13} = \sin\theta\sin\phi\partial_u + (-\csc\theta\cos\phi + \cos\phi\tan\theta\cos\theta)\partial_r$$
$$- \frac{1}{r}\cos\theta\sin\phi\partial_\theta - \frac{1}{r}\csc\theta\cos\phi\partial_\phi, \quad g = 0,$$
$$X_{14} = \sin\theta\cos\phi\partial_u - (\csc\theta\cos\phi + \tan\theta\cos\theta\cos\phi)\partial_r - \frac{1}{r}\cos\theta\cos\phi\partial_\theta$$
$$+ \frac{1}{r}\csc\theta\sin\phi\partial_\phi, \quad g = 0,$$
$$X_{15} = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \quad g = 0,$$
$$X_{16} = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi, \quad g = 0,$$
$$X_{17} = \partial_\phi, \quad g = 0.$$

Notes. From this list, we can easily conclude that $\{X_i : i = 8\ldots17\}$ form the 10-dimensional algebra of isometries which with $n = 4$ corresponds to the maximal $\frac{1}{2}n(n+1) =$ 10-dimensional algebra. That is, the respective manifold is isometric to one of the (a) 4-dimensional Euclidean space, (b) 4-dimensional sphere, (c) 4-dimensional projective space or (d) 4-dimensional simply connected hyperbolic space (see [7]). This confirms the known result that $m = 0$ is equivalent to the Minkowski metric. Each lead to conserved quantities from Noether's theorem [8, 9]. As a sample case, from X_{15}, we get

$$\begin{aligned} T^{15} &= L\sigma + (\eta - \dot{u}\sigma)\frac{\partial L}{\partial \dot{u}} + (\rho - \dot{r}\sigma)\frac{\partial L}{\partial \dot{r}} + (\tau - \dot{\theta}\sigma)\frac{\partial L}{\partial \dot{\theta}} + (\kappa - \dot{\phi}\sigma)\frac{\partial L}{\partial \dot{\phi}} - g \\ &= 2r^2\dot{\theta}\cos\phi - 2r^2\dot{\phi}\cot\theta\sin^3\theta \end{aligned} \tag{2.3}$$

which, incidentally, lead to two of the four Euler-Lagrange equations. The remaining symmetries $\{X_i : i = 1\ldots7\}$, lead to seven new (previously unknown) conserved quantities. For e.g., from X_6 we get

$$\begin{aligned} T^6 = s(-\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin\theta^2\dot{\phi}^2) + \left(\frac{1}{2}u - s\dot{u}\right)(-2\dot{u} - 2\dot{r}) \\ - (r - 2s\dot{r})\dot{u} - 2s\dot{\theta}^2 r^2 - 2s\dot{\phi}^2 r^2\sin\theta^2 \end{aligned} \tag{2.4}$$

whose total divergence lead to the complete system of Euler-Lagrange equations.

(ii) Case $m = k$, constant:

$$\begin{aligned} &X_1 = \partial_s, \quad g = 0, \\ &X_2 = \partial_u, \quad g = 0, \\ &X_3 = \partial_\phi, \quad g = 0, \\ &X_4 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0, \\ &X_5 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0. \end{aligned}$$

Note. Here, $\{X_i : i = 2\ldots5\}$ is the algebra of isometries rendering the metric equivalent to the Schwarzschild metric.

(iii) Case $m = u$:

$$\begin{aligned} &X_1 = \partial_s, \quad g = 0, \\ &X_2 = \partial_\phi, \quad g = 0, \\ &X_3 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0, \\ &X_4 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0, \\ &X_5 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = 0. \end{aligned}$$

Here, we have interesting situation that, $\{X_i : i = 2\ldots4\}$ form the 3-dimensional algebra of isometries with X_5 providing an extra nontrivial conservation law, viz.

$$T^5 = s\left(-\left(1-\frac{2u}{r}\right)\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin\theta^2\dot{\phi}^2\right)$$
$$+\left(\frac{1}{2}u - s\dot{u}\right)\left(-2\dot{u}\left(1-\frac{2u}{r}\right) - 2\dot{r}\right)$$
$$-(r-2s\dot{r})\dot{u} - 2s\dot{\theta}^2r^2 - 2s\dot{\phi}^2r^2\sin\theta^2. \qquad (2.5)$$

(iv) Case $m = m(u)$, arbitrary function of u:

$$X_1 = \partial_s, \quad g = 0,$$
$$X_2 = \partial_\phi, \quad g = 0,$$
$$X_3 = -\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi, \quad g = 0,$$
$$X_4 = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \quad g = 0.$$

We have, here, a similar conclusion regarding the algebra of isometries as in (iii) but the additional conservation law is lost.

3 Noether Symmetries of the Metric (1.2)

For the second metric (1.2), we only consider the Noether symmetries and conclude the subalgebra of isometries. It will be clear that Lie algebra of point symmetries of the geodesic equations will be as large as above. Here the Lagrangian is

$$L = -\left(1-\frac{2m(u)}{r}\right)\dot{u}^2 - \frac{4m(u)}{r}\dot{u}\dot{r} + \left(1+\frac{2m(u)}{r}\right)\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \qquad (3.1)$$

whose corresponding Noether symmetries separate into the same following cases.

(i) Case $m = 0$:

$$X_1 = \frac{1}{2}s^2\partial_s + \frac{1}{2}su\partial_u + \frac{1}{2}rs\partial_r, \quad g = \frac{1}{2}r^2 - \frac{1}{2}u^2,$$
$$X_2 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = \frac{1}{2}u + r,$$
$$X_3 = \partial_s, \quad g = 0,$$
$$X_4 = -\frac{1}{2}s\partial_u, \quad g = u,$$
$$X_5 = \frac{1}{2}s(-\cot\theta\cos\theta\sin\phi + \csc\theta\sin\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\sin\phi\,\partial_\theta$$
$$+\frac{1}{2}\frac{s}{r}\csc\theta\cos\phi\,\partial_\phi, \quad g = r\sin\theta\sin\phi,$$
$$X_6 = \frac{1}{2}s(-\cot\theta\cos\theta\cos\phi + \csc\theta\cos\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\cos\phi\,\partial_\theta$$
$$+\frac{1}{2}\frac{s}{r}\csc\theta\sin\phi\,\partial_\phi, \quad g = r\sin\theta\cos\phi,$$

$$X_7 = \frac{1}{2} s \cos\theta \partial_r - \frac{1}{2}\frac{s}{r} \sin\theta \partial_\theta, \quad g = r\cos\theta,$$

$$X_8 = -r\sin\theta\cos\phi\partial_u + u\cot\theta\cos\theta\cos\phi - u\csc\theta\cos\phi\partial_r - \frac{u}{r}\cos\theta\cos\phi\partial_\theta$$
$$+ \frac{u}{r}\csc\theta\sin\phi\partial_\phi, \quad g = 0,$$

$$X_9 = r\sin\theta\sin\phi\partial_u - u\cot\theta\cos\theta\sin\phi + u\csc\theta\sin\phi\partial_r - \frac{u}{r}\cos\theta\sin\phi\partial_\theta$$
$$+ \frac{1}{r}\csc\theta\cos\phi\partial_\phi, \quad g = 0,$$

$$X_{10} = -\csc\theta\cos\phi + \cot\theta\cos\theta\cos\phi\partial_r - \frac{u}{r}\cos\theta\cos\phi\partial_\theta + \frac{1}{r}\csc\theta\sin\phi\partial_\phi, \quad g = 0,$$

$$X_{11} = -\csc\theta\sin\phi + \cot\theta\cos\theta\sin\phi\partial_r + \frac{u}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{r}\csc\theta\cos\phi\partial_\phi, \quad g = 0,$$

$$X_{12} = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0,$$

$$X_{13} = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0,$$

$$X_{14} = \partial_\phi, \quad g = 0,$$

$$X_{15} = -r\cos\theta\partial_u - u\cos\theta\partial_r + \frac{u}{r}\sin\theta\partial_\theta, \quad g = 0,$$

$$X_{16} = -\cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta, \quad g = 0,$$

$$X_{17} = \partial_u, \quad g = 0.$$

(ii) Case $m = k$, constant:

$$X_1 = \partial_s, \quad g = 0,$$
$$X_2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0,$$
$$X_3 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0,$$
$$X_4 = \partial_\phi, \quad g = 0,$$
$$X_5 = \partial_u, \quad g = 0.$$

(iii) Case $m = u$:

$$X_1 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = 0,$$
$$X_2 = \partial_s, \quad g = 0,$$
$$X_3 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0,$$
$$X_4 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0,$$
$$X_5 = \partial_\phi, \quad g = 0.$$

(iv) Case $m = m(u)$:

$$X_1 = \partial_s, \quad g = 0,$$

$$X_2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0,$$

$$X_3 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0,$$

$$X_4 = \partial_\phi, \quad g = 0.$$

4 Discussion and Conclusion

We have shown that a large amount information can be extracted from a knowledge of the vector fields (one parameter Lie group transformations) that leave the action integral invariant. In addition to a larger class of conservation laws than those given by the isometries or Killing vectors, we can conclude what the isometries actually are and that these form a Lie subalgebra of the Noether symmetry algebra. We have performed the calculations on some versions of the Vaidiya metric yielding some previously unknown information regarding the corresponding manifold. Lastly, with particular reference to this metric, we concluded that the only variations on $m(u)$ that occur are $m = 0$, $m =$ constant, $m = u$ and $m = m(u)$.

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